

# Blow-up and symmetry of sign-changing solutions to some critical elliptic equations

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Received 22 December 2005; revised 18 May 2006

Available online 5 July 2006

## Abstract

In this paper we continue the analysis of the blow-up of low energy sign-changing solutions of semi-linear elliptic equations with critical Sobolev exponent, started in [M. Ben Ayed, K. El Mehdi, F. Pacella, Blow-up and nonexistence of sign-changing solutions to the Brezis–Nirenberg problem in dimension three, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, in press]. In addition we prove axial symmetry results for the same kind of solutions in a ball.

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MSC: 35J20; 35J60

Keywords: Blow-up analysis; Sign-changing solutions; Nodal domains; Critical exponent

## 1. Introduction

In this paper we study low energy sign-changing solutions of the following semi-linear elliptic problem:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

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<sup>1</sup> Supported by "l'Agence Universitaire Francophone."

<sup>2</sup> Supported by MURST, Project "Metodi Variazionali ed Equazioni Differenziali non lineari."

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $\lambda$  is a positive real parameter and  $2^* = 2n/(n-2)$  is the critical Sobolev exponent for the embedding of  $H_0^1(\Omega)$  into  $L^{2^*}(\Omega)$ .

Problem (1) is known as the Brezis–Nirenberg problem since the first fundamental results about the existence of positive solutions were obtained in the celebrated paper [7], by Brezis and Nirenberg. In this paper it was also enlightened the crucial role played by the dimension in the study of (1). Indeed if the dimension  $n$  is larger than 3, in any bounded domain  $\Omega$ , there exist positive solutions of (1), for every  $\lambda \in (0, \lambda_1(\Omega))$ ,  $\lambda_1(\Omega)$  being the first eigenvalue of  $-\Delta$  on  $\Omega$  with zero Dirichlet boundary conditions. Instead if  $n = 3$  in any strictly starshaped domains, there are no positive solutions for  $\lambda$  close to zero. The reason of this difference relies on the presence in the equation of the lower order term  $\lambda u$  which makes estimates quite different. This obstacle reflects on the study of the asymptotic behavior of the solutions as well as on the investigation of the existence of sign-changing solutions. Let us mention that also the case  $n = 4$  presents more difficulties compared to the higher dimensions. Indeed while several results are available for the case  $n \geq 5$  [9,12] not many results exist for  $n = 4$  and even less for  $n = 3$ .

In [6] we investigated the asymptotic behavior of low energy sign-changing solutions of (1) in dimension 3, as the parameter  $\lambda$  goes to the limit parameter

$$\bar{\lambda}(\Omega) = \inf\{\lambda \in \mathbb{R}: (1) \text{ has a sign-changing solution } u_\lambda \text{ with } \|u_\lambda\|_\lambda^2 \leq 2S^{3/2}\}, \quad (1.1)$$

where  $\|u\|_\lambda^2 = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2$  and  $S$  is the best Sobolev constant for the embedding of  $H_0^1(\Omega)$  into  $L^{2^*}(\Omega)$ . More precisely we proved that if  $\bar{\lambda}(\Omega) < \lambda_1(\Omega)$  and  $(u_\lambda)$  is a family of sign-changing solutions of (1), for  $n = 3$ , such that  $\int_\Omega |\nabla u_\lambda|^2 \rightarrow 2S^{3/2}$  and  $u_\lambda \rightarrow 0$  as  $\lambda \rightarrow \bar{\lambda}(\Omega)$  (which always is the case of any minimizing sequence, for (1.1), if  $\bar{\lambda}(\Omega)$  is not achieved) then as  $\lambda \rightarrow \bar{\lambda}(\Omega)$  the solution  $u_\lambda$  blows up at exactly two points which are the limit of the concentration points of the positive and negative part of  $u_\lambda$  and whose distance, from each other and from the boundary, is bounded by a positive constant depending only on  $\Omega$ . Moreover, for  $\lambda$  close to  $\bar{\lambda}(\Omega)$ ,  $u_\lambda$  has only two nodal regions and the concentration speeds of the positive and negative part are comparable. We refer the reader to [6, Theorems 3.1 and 4.1] for the precise statements.

Though the case  $n = 3$  is usually considered more difficult and results true in dimension 3 are expected to hold also in higher dimensions with similar or even simpler proofs, the proof of [6] only works in dimension 3, because in applying Pohozaev's identity and getting the convergence of certain integrals, the role of the dimension is crucial.

The first part of this paper is devoted to describe the asymptotic behavior, as  $\lambda \rightarrow 0$ , of sign-changing solutions of (1) whose energy converges to the value  $2S^{n/2}$ , in higher dimensions. The method used here is different from that of [6] and while some initial results hold for any dimension  $n \geq 3$  without any extra assumptions, a more precise blow-up analysis requires the a priori assumption that the concentration speeds of the two concentration points are comparable, while in [6] this result was derived without any extra hypothesis.

More precisely we have:

**Theorem 1.1.** *Let  $n \geq 3$  and  $(u_\lambda)$  be a family of sign-changing solutions of (1) satisfying*

$$\|u_\lambda\|^2 := \int_\Omega |\nabla u_\lambda|^2 \rightarrow 2S^{n/2}, \quad \text{as } \lambda \rightarrow 0 \text{ if } n \geq 4 \text{ or } \lambda \rightarrow \bar{\lambda}(\Omega) \text{ if } n = 3, \quad (1.2)$$

*and, if  $n = 3$ , we also require that  $u_\lambda \rightarrow 0$  as  $\lambda \rightarrow \bar{\lambda}(\Omega)$ .*

Then, the set  $\Omega \setminus \{x \in \Omega: u_\lambda(x) = 0\}$  has exactly two connected components.

Furthermore, there exist two points  $a_{\lambda,1}$  and  $a_{\lambda,2}$  in  $\Omega$  (one of them is the global maximum point of  $|u_\lambda|$ ) and two positive reals  $\mu_{\lambda,1}$  and  $\mu_{\lambda,2}$  such that, as  $\lambda \rightarrow 0$  if  $n \geq 4$ , and  $\lambda \rightarrow \bar{\lambda}(\Omega)$  if  $n = 3$ , we have

$$\|u_\lambda - P\delta_{(a_{\lambda,1}, \mu_{\lambda,1})} + P\delta_{(a_{\lambda,2}, \mu_{\lambda,2})}\| \rightarrow 0, \quad \mu_{\lambda,i} d(a_{\lambda,i}, \partial\Omega) \rightarrow +\infty, \text{ for } i \in \{1, 2\}, \quad (1.3)$$

where  $P\delta_{(a,\mu)}$  denotes the projection of  $\delta_{(a,\mu)}$  on  $H_0^1(\Omega)$ , that is,

$$\Delta P\delta_{(a,\mu)} = \Delta\delta_{(a,\mu)} \quad \text{in } \Omega, \quad P\delta_{(a,\mu)} = 0 \quad \text{on } \partial\Omega \quad \text{and}$$

$$\delta_{(a,\mu)}(x) = \frac{\beta_n \mu^{(n-2)/2}}{(1 + \mu^2|x - a|^2)^{(n-2)/2}}.$$

Here  $\beta_n$  is a positive constant chosen so that  $-\Delta\delta_{(a,\mu)} = \delta_{(a,\mu)}^{(n+2)/(n-2)} (\beta_n(n(n-2))^{(n-2)/4})$ .

Moreover, if we assume that  $-\max u_\lambda / \min u_\lambda$  is bounded above and below then  $a_{\lambda,1}$  and  $a_{\lambda,2}$  are two global extremum points of  $u_\lambda$  and we have

$$\mu_{\lambda,i} = (|u_\lambda(a_{\lambda,i})|/\beta_n)^{2/(n-2)}, \quad \text{for } i \in \{1, 2\}.$$

Now, we need to obtain some information about the concentration points and the concentration speeds. To this aim, we introduce some notations. We denote by  $G$  the Green's function of the Laplace operator defined by:  $\forall x \in \Omega$

$$-\Delta G(x, \cdot) = c_n \delta_x \quad \text{in } \Omega, \quad G(x, \cdot) = 0 \quad \text{on } \partial\Omega,$$

where  $\delta_x$  is the Dirac mass at  $x$  and  $c_n = (n-2)\omega_n$ , with  $\omega_n$  denoting the area of the unit sphere in  $\mathbb{R}^n$ . We denote by  $H$  the regular part of  $G$ , that is,

$$H(x_1, x_2) = |x_1 - x_2|^{2-n} - G(x_1, x_2) \quad \text{for } (x_1, x_2) \in \Omega^2.$$

For  $x = (x_1, x_2) \in \Omega^2 \setminus \Gamma$ , with  $\Gamma = \{(y, y)/y \in \Omega\}$ , we denote by  $M(x)$  the matrix defined by

$$M(x) = (m_{ij})_{1 \leq i, j \leq 2}, \quad \text{where } m_{ii} = H(x_i, x_i), \quad m_{12} = m_{21} = G(x_1, x_2). \quad (1.4)$$

Then we have:

**Theorem 1.2.** Let  $n \geq 4$  and let  $(u_\lambda)$  be a family of sign-changing solutions of (1) satisfying (1.2). If there exists a positive constant  $\gamma$  such that

$$\gamma^{-1} \leq -\max u_\lambda (\min u_\lambda)^{-1} \leq \gamma, \quad (1.5)$$

then each concentration point  $a_{\lambda,i}$ , defined in Theorem 1.1, converges to  $\bar{a}_i \in \Omega$ , for  $i = 1, 2$  with  $\bar{a}_1 \neq \bar{a}_2$ , and the concentration speeds  $\mu_{\lambda,i}$ , defined in Theorem 1.1, satisfy

$$(c_2 \lambda / c_1)^{(n-2)/(2n-8)} \mu_{\lambda,i}^{(n-2)/2} \rightarrow \bar{A}_i^{-1}, \quad \text{as } \lambda \rightarrow 0 \text{ if } n \geq 5, \quad (1.6)$$

$$(c_3 / c_1) \lambda \log(\mu_{\lambda,i}) \rightarrow \bar{A}, \quad \text{as } \lambda \rightarrow 0 \text{ for } i = 1, 2, \text{ if } n = 4, \quad (1.7)$$

where  $\bar{\Lambda}_i$  and  $\bar{\Lambda}$  are positive constants,  $c_1 = \beta_n \int_{\mathbb{R}^n} \delta_{(0,1)}^{(n+2)/(n-2)}$ ,  $c_2 = \frac{n-2}{2} \beta_n^2 \int_{\mathbb{R}^n} \frac{(|y|^2-1)dy}{(1+|y|^2)^{n-1}}$  and  $c_3 = \beta_4^2 \omega_4$  ( $\omega_4$  is the area of the unit sphere of  $\mathbb{R}^4$ ).

Furthermore, if  $n \geq 5$ ,  $(\bar{a}_1, \bar{a}_2, \bar{\Lambda}_1, \bar{\Lambda}_2)$  is a critical point of the function

$$\Psi : \Omega^2 \setminus \Gamma \times (0, \infty)^2 \rightarrow \mathbb{R}; \quad (a, \Lambda) := (a_1, a_2, \Lambda_1, \Lambda_2) \mapsto {}^t \Lambda M(a) \Lambda - \Lambda_1^{\frac{4}{n-2}} - \Lambda_2^{\frac{4}{n-2}},$$

where  $M(a)$  is the matrix defined by (1.4).

If  $n = 4$ , denoting by  $\bar{\gamma}_i$  the limit of  $\mu_{\lambda,i}/\mu_{\lambda,j}$  ( $\bar{\gamma}_1 = \bar{\gamma}_2^{-1}$ ), up to a subsequence, then  $(\bar{\gamma}_i, \bar{\Lambda})$  satisfies

$$H(\bar{a}_i, \bar{a}_i) + \bar{\gamma}_i G(\bar{a}_1, \bar{a}_2) - \bar{\Lambda} = 0 \quad \text{and} \quad \frac{\partial H(\bar{a}_i, \bar{a}_i)}{\partial \bar{a}_i} + 2\bar{\gamma}_i \frac{\partial G(\bar{a}_1, \bar{a}_2)}{\partial \bar{a}_i} = 0. \quad (1.8)$$

**Remark 1.3.** Let  $x_0, y_0$  be such that  $x_0 = \bar{\gamma}_2 y_0$  and  $\bar{\Lambda} = e^{(x_0^2 + y_0^2)}$ , then (1.8) implies that  $(\bar{a}_1, \bar{a}_2, x_0, y_0)$  is a critical point of the function

$$\Psi_1 : \Omega^2 \setminus \Gamma \times (0, \infty)^2 \rightarrow \mathbb{R}; \quad (a, \Lambda) := (a_1, a_2, x, y) \mapsto {}^t \Lambda M(a) \Lambda - e^{x^2 + y^2}.$$

Next, we describe the asymptotic behavior, as  $\lambda \rightarrow 0$ , of low energy sign-changing solutions of (1) outside the limit concentration points.

**Theorem 1.4.** Let  $n \geq 4$  and let  $(u_\lambda)$  be a family of sign-changing solutions of (1) satisfying (1.2) and (1.5). Then the limit concentration points  $\bar{a}_1$  and  $\bar{a}_2$ , defined in Theorem 1.2, are two isolated simple blow-up points of  $(u_\lambda)$  (see [11] for definitions). Moreover, there exist positive constants  $m_1$  and  $m_2$  such that

$$\begin{aligned} \lambda^{\frac{2-n}{2n-8}} u_\lambda &\rightarrow m_1 G(\bar{a}_1, \cdot) - m_2 G(\bar{a}_2, \cdot) \quad \text{in } C_{\text{loc}}^2(\Omega \setminus \{\bar{a}_1, \bar{a}_2\}), \quad \text{as } \lambda \rightarrow 0 \text{ if } n \geq 5, \\ |u_\lambda|_\infty u_\lambda &\rightarrow m_1 G(\bar{a}_1, \cdot) - m_2 G(\bar{a}_2, \cdot) \quad \text{in } C_{\text{loc}}^2(\Omega \setminus \{\bar{a}_1, \bar{a}_2\}), \quad \text{as } \lambda \rightarrow 0 \text{ if } n = 4. \end{aligned}$$

**Remark 1.5.** For the result analogous to Theorem 1.4 in dimension 3, we refer the reader to [6, (4.10)].

The second part of the paper is devoted to deduce symmetry properties of low energy sign-changing solutions of (1) when  $\Omega$  is a ball exploiting the blow-up analysis carried out, together with the method used in [8].

As it will be clear from the proof, to get the symmetry result it is important to know that  $u_\lambda$  concentrates only at two points whose distance, from each other and from the boundary, is bounded away from zero and the concentration speeds are comparable.

Hence, to state the symmetry result we consider a family  $(u_\lambda)$  of sign-changing solutions which, as  $\lambda \rightarrow 0$  if  $n \geq 4$ , or  $\lambda \rightarrow \bar{\lambda}(\Omega)$  if  $n = 3$ , satisfies the following assumptions: denoting by  $a_{\lambda,1}$  the maximum point and by  $a_{\lambda,2}$  the minimum point,

- (1)  $\|u_\lambda\|^2 \rightarrow 2S^{n/2}$ ,
- (2)  $\|u_\lambda - P\delta_{(a_{\lambda,1}, \mu_{\lambda,1})} + P\delta_{(a_{\lambda,2}, \mu_{\lambda,2})}\| \rightarrow 0$ ,
- (3)  $u_\lambda(a_{\lambda,i})^2 \rightarrow +\infty, i = 1, 2$ ,

- (4)  $\exists \alpha > 0$  such that  $d(a_{\lambda,i}, \partial\Omega) \geq \alpha$ ,  $i = 1, 2$ ,  $|a_{\lambda,1} - a_{\lambda,2}| \geq \alpha$ ,  
(5)  $\exists \bar{c} > 0$  such that  $\bar{c}^{-1} \leq -u_{\lambda}(a_{\lambda,1})u_{\lambda}(a_{\lambda,2})^{-1} \leq \bar{c}$ .

Obviously the solutions considered in [6] in dimension 3 and the ones satisfying the hypotheses of Theorem 1.2 for  $n \geq 4$  verify assumptions (1)–(5).

**Theorem 1.6.** *Assume that  $n \geq 3$ . Let  $\Omega$  be a ball and  $(u_{\lambda})$  be a family of sign-changing solutions of (1) satisfying the assumptions of Theorem 1.1 and, if  $n \geq 4$ , we also require that  $(u_{\lambda})$  satisfies (1.5). Then, for  $\lambda$  close to zero if  $n \geq 4$  or  $\lambda$  close to  $\bar{\lambda}(\Omega)$  if  $n = 3$ , the concentration points  $a_{\lambda,1}$  and  $a_{\lambda,2}$  of  $u_{\lambda}$ , given by Theorem 1.1 if  $n \geq 4$  and by [6, Theorem 3.1] for  $n = 3$ , lay on the same line passing through the origin and  $u_{\lambda}$  is axially symmetry with respect to this line. Moreover, if  $n \geq 4$ ,  $a_{\lambda,1}$  and  $a_{\lambda,2}$  lay on different sides with respect to  $T$ , where  $T$  is any hyperplane passing through the origin but not containing  $a_{\lambda,1}$ .*

*In addition, all the critical points of  $u_{\lambda}$  belong to the symmetry axis and*

$$\frac{\partial u_{\lambda}}{\partial \nu_T}(x) > 0 \quad \forall x \in T \cap \Omega, \quad (1.9)$$

where  $\nu_T$  is the normal to  $T$ , oriented towards the half-space containing  $a_{\lambda,1}$ .

**Remark 1.7.** The kind of symmetry proved in Theorem 1.6, i.e., the axial symmetry together with the monotonicity property (1.9) is often called “foliated Schwartz symmetry.”

The outline of the paper is the following. Section 2 is devoted to the proof of Theorem 1.1. In Section 3, we give some useful estimates and we prove Theorems 1.2 and 1.4. Finally, we prove Theorem 1.6 in Section 4.

## 2. Proof of Theorem 1.1

For  $n = 3$ , the theorem is proved in [6]. It remains to see what happens for  $n \geq 4$ . Regarding the connected components of  $\Omega \setminus \{x \in \Omega: u_{\lambda}(x) = 0\}$ , let  $\Omega_1$  be one of them. Multiplying (1) by  $u_{\lambda}$  and integrating on  $\Omega_1$ , we derive that

$$\int_{\Omega_1} |\nabla u_{\lambda}|^2 \geq S^{n/2}(1 + o(1)), \quad (2.1)$$

where we have used the Sobolev embedding and the fact that  $\lambda \rightarrow 0$  and  $\lambda_1(\Omega_1) \int_{\Omega_1} u_{\lambda}^2 \leq \int_{\Omega_1} |\nabla u_{\lambda}|^2$ , where  $\lambda_1(\Omega_1)$  is the first Dirichlet eigenvalue of  $-\Delta$  on  $\Omega_1$ .

Since  $\|u_{\lambda}\|^2 \rightarrow 2S^{n/2}$ , we deduce that there are only two connected components.

The following lemma shows that the energy of the solution  $u_{\lambda}$  converges to  $S^{n/2}$  in each connected component. In fact, we have:

**Lemma 2.1.** *Let  $n \geq 4$  and let  $(u_{\lambda})$  be a family of sign-changing solutions of (1) satisfying (1.2). Then*

- (i)  $\int_{\Omega} |\nabla u_{\lambda}^+|^2 \rightarrow S^{n/2}, \quad \int_{\Omega} |\nabla u_{\lambda}^-|^2 \rightarrow S^{n/2}, \quad \text{as } \lambda \rightarrow 0,$
- (ii)  $\int_{\Omega} (u_{\lambda}^+)^{\frac{2n}{n-2}} \rightarrow S^{n/2}, \quad \int_{\Omega} (u_{\lambda}^-)^{\frac{2n}{n-2}} \rightarrow S^{n/2}, \quad \text{as } \lambda \rightarrow 0,$
- (iii)  $u_{\lambda} \rightharpoonup 0, \quad \text{as } \lambda \rightarrow 0,$
- (iv)  $M_{\lambda,+} := \max_{\Omega} u_{\lambda}^+ \rightarrow +\infty, \quad M_{\lambda,-} := \max_{\Omega} u_{\lambda}^- \rightarrow +\infty, \quad \text{as } \lambda \rightarrow 0,$

where  $u_{\lambda}^+ = \max(u_{\lambda}, 0)$  and  $u_{\lambda}^- = \max(0, -u_{\lambda})$ .

**Proof.** Recall that  $\Omega$  has only two connected components. Hence, claim (i) follows from (2.1) and the fact that  $\|u_{\lambda}\|^2 = \|u_{\lambda}^+\|^2 + \|u_{\lambda}^-\|^2 \rightarrow 2S^{n/2}$ , as  $\lambda \rightarrow 0$ .

Now, multiplying (1) by  $u_{\lambda}^{\pm}$  and integrating on  $\Omega$ , we obtain

$$\int_{\Omega} |\nabla u_{\lambda}^{\pm}|^2 = \int_{\Omega} (u_{\lambda}^{\pm})^{\frac{2n}{n-2}} + \lambda \int_{\Omega} (u_{\lambda}^{\pm})^2. \quad (2.2)$$

Therefore, claim (ii) follows from (2.2) and claim (i).

Regarding (iii), it follows from (i), (ii) and the fact that

$$S = \inf \{ \|u\|^2 |u|_{L^{2n/(n-2)}}^{-2}, \quad u \neq 0, \quad u \in H_0^1(\Omega) \} \quad (2.3)$$

is not achieved if  $\Omega \neq \mathbb{R}^n$ .

It remains to prove claim (iv). Arguing by contradiction, we assume that  $M_{\lambda,\pm} \leq c$  as  $\lambda \rightarrow 0$ . Therefore  $u_{\lambda}^{\pm} \in L^{\infty}(\Omega)$  and  $u_{\lambda}^{\pm} \rightarrow 0$  a.e. Thus  $(u_{\lambda}^{\pm})^{2n/(n-2)} \rightarrow 0$  in  $L^1(\Omega)$  which contradicts claim (ii). Hence  $M_{\lambda,\pm} \rightarrow \infty$  as  $\lambda \rightarrow 0$ . Therefore our lemma is proved.  $\square$

Observe that (i) and (ii) imply that  $(u_{\lambda}^+)$  and  $(u_{\lambda}^-)$  are two positive minimizing sequences for the functional

$$J(u) := \|u\|^2 |u|_{L^{2n/(n-2)}}^{-2}, \quad u \in H_0^1(\Omega) \setminus \{0\}.$$

By easy computations we prove that  $\nabla J(u_{\lambda}^{\pm}) \rightarrow 0$  as  $\lambda \rightarrow 0$ . Hence  $(u_{\lambda}^{\pm})$  are two positive Palais–Smale sequences. Now using [16] and the fact that  $S$  is not achieved in  $\Omega$ , we obtain that there exist  $a_{\lambda,1}, a_{\lambda,2}, \mu_{\lambda,1}$  and  $\mu_{\lambda,2}$  such that

$$\begin{aligned} \|u_{\lambda}^+ - P\delta_{(a_{\lambda,1}, \mu_{\lambda,1})}\| &\rightarrow 0, & \|u_{\lambda}^- - P\delta_{(a_{\lambda,2}, \mu_{\lambda,2})}\| &\rightarrow 0, \\ \mu_{\lambda,i} d(a_{\lambda,i}, \partial\Omega) &\rightarrow +\infty, & \text{for } i = 1, 2, \end{aligned}$$

which imply (1.3).

To prove that one of the points is the global maximum point of  $|u_{\lambda}|$ , we need to study carefully the asymptotic behavior of  $u_{\lambda}$ . We start by:

**Lemma 2.2.** Let  $a_\lambda$  be such that  $|u_\lambda(a_\lambda)|/\max |u_\lambda| \geq c$ , where  $c$  is a positive constant independent of  $\lambda$ . Then we have

$$|u_\lambda(a_\lambda)|^{2/(n-2)} d(a_\lambda, \partial\Omega) \not\rightarrow 0, \quad \text{as } \lambda \rightarrow 0.$$

**Proof.** Without loss of generality, we can assume that  $M_\lambda := u_\lambda(a_\lambda) > 0$ . Arguing by contradiction, we assume that  $M_\lambda^{2/(n-2)} d(a_\lambda, \partial\Omega) \rightarrow 0$  as  $\lambda \rightarrow 0$ . Let

$$\tilde{u}_\lambda(X) := M_\lambda^{-1} u_\lambda(a_\lambda + X/M_\lambda^{2/(n-2)}), \quad \text{for } X \in \tilde{\Omega}_\lambda := M_\lambda^{2/(n-2)} (\Omega - a_\lambda). \quad (2.4)$$

Observe that  $|\tilde{u}_\lambda|$  is bounded in  $\tilde{\Omega}_\lambda$ . Let  $D := B(0, 1) \cap \tilde{\Omega}_\lambda$ . For  $y \in B(0, 1/4) \cap \tilde{\Omega}_\lambda$ , we have

$$\tilde{u}_\lambda(y) = c \int_D G_D(x, y) (-\Delta \tilde{u}_\lambda)(x) dx - c \int_{\partial D} \frac{\partial G_D}{\partial \nu_x}(x, y) \tilde{u}_\lambda(x) dx,$$

where  $G_D$  denotes the Green function defined in  $D^2$ . Thus we derive

$$\nabla_y \tilde{u}_\lambda(y) = c \int_D \nabla_y G_D(x, y) (-\Delta \tilde{u}_\lambda)(x) dx - c \int_{\partial D} \nabla_y \frac{\partial G_D}{\partial \nu_x}(x, y) \tilde{u}_\lambda(x) dx. \quad (2.5)$$

Since  $\tilde{u}_\lambda$  is bounded in  $D$  and  $|\nabla_y G_D(x, y)| \leq c|x - y|^{1-n}$ , it follows that the first integral in (2.5) is bounded. For the other one, we have  $\partial D = (\partial B(0, 1) \cap \tilde{\Omega}_\lambda) \cup (B(0, 1) \cap \partial \tilde{\Omega}_\lambda)$ , and since  $\tilde{u}_\lambda = 0$  on  $\partial \tilde{\Omega}_\lambda$ , the second integral is computed only on  $\partial B(0, 1) \cap \tilde{\Omega}_\lambda$ . In this set we have  $|x - y| \geq c$  for each  $y \in B(0, 1/4) \cap D$ , therefore  $|\nabla_y \partial G_D(x, y)/\partial \nu_x|$  is bounded. Thus the second integral in (2.5) is also bounded. Hence  $|\nabla_y \tilde{u}_\lambda(y)| \leq c$  for each  $y \in B(0, 1/4) \cap D$ . To conclude, without loss of generality, we can assume that  $\{t\tilde{a} : 0 \leq t < 1\} \subset \tilde{\Omega}_\lambda$ , where  $\tilde{a}$  belongs to  $\partial \tilde{\Omega}_\lambda$  and satisfies  $|\tilde{a}| = d(0, \partial \tilde{\Omega}_\lambda) = M_\lambda^{2/(n-2)} d(a_\lambda, \partial\Omega)$ , and we write

$$\tilde{u}_\lambda(0) = \tilde{u}_\lambda(\tilde{a}) - \nabla \tilde{u}_\lambda(t_1 \tilde{a}) \tilde{a}, \quad \text{for } t_1 \in (0, 1).$$

Since  $|\nabla \tilde{u}_\lambda(t_1 \tilde{a})|$  is bounded,  $\tilde{u}_\lambda(\tilde{a}) = 0$ ,  $\tilde{u}_\lambda(0) = 1$ , and we have assumed that  $|\tilde{a}|$  goes to zero, we get a contradiction. This implies that  $M_\lambda^{2/(n-2)} d(a_\lambda, \partial\Omega) \not\rightarrow 0$  as  $\lambda \rightarrow 0$ . Hence our lemma follows.  $\square$

To complete the proof of Theorem 1.1, let  $M_{\lambda,+} := u_\lambda(a_{\lambda,+}) := \max u_\lambda$  and  $M_{\lambda,-} := -u_\lambda(a_{\lambda,-}) := -\min u_\lambda$  and, without loss of generality, we can assume that  $M_{\lambda,+} \geq M_{\lambda,-}$ . Let us define

$$\tilde{u}_\lambda(X) := M_{\lambda,+}^{-1} u_\lambda(a_{\lambda,+} + X/M_{\lambda,+}^{2/(n-2)}), \quad \text{for } X \in \tilde{\Omega}_\lambda : M_{\lambda,+}^{2/(n-2)} (\Omega - a_{\lambda,+}).$$

By Lemma 2.1, the limit domain of  $\tilde{\Omega}_\lambda$ , denoted by  $\Pi$ , has to be the whole space  $\mathbb{R}^n$  or a half-space and by Lemma 2.2, it contains the origin. Since  $\tilde{u}_\lambda$  is bounded in  $\tilde{\Omega}_\lambda$ , using the standard elliptic theory, it converges in  $C_{\text{loc}}^2(\Pi)$  to a function  $u$  satisfying

$$-\Delta u = |u|^{2^*-2} u \quad \text{in } \Pi, \quad u(0) = 1, \quad u = 0 \quad \text{on } \partial \Pi, \quad \text{and} \quad \int_\Pi |\nabla u|^2 \leq 2S^{n/2}. \quad (2.6)$$

Observe that, any sign-changing solution  $w$  of (2.6) satisfies  $\|w\|^2 > 2S^{n/2}$ . Thus we derive that  $u$  is positive. It follows that  $\Pi$  has to be  $\mathbb{R}^n$  and  $u = \delta_{(0, \beta_n^{2/(2-n)})}$ . Hence

$$\|u_\lambda^+ - P\delta_{(a_{\lambda,+}, (M_{\lambda,+}/\beta_n)^{2/(n-2)})}\| \rightarrow 0 \quad \text{and} \quad M_{\lambda,+}^{2/(n-2)} d(a_{\lambda,+}, \partial\Omega) \rightarrow \infty, \quad \text{as } \lambda \rightarrow 0.$$

Finally, if we assume that  $M_{\lambda,+}/M_{\lambda,-} \leq c$ , then we can repeat the above argument and we prove that

$$\|u_\lambda^- - P\delta_{(a_{\lambda,-}, (M_{\lambda,-}/\beta_n)^{2/(n-2)})}\| \rightarrow 0 \quad \text{and} \quad M_{\lambda,-}^{2/(n-2)} d(a_{\lambda,-}, \partial\Omega) \rightarrow \infty, \quad \text{as } \lambda \rightarrow 0.$$

The proof of our theorem is thereby completed.

### 3. Proof of Theorems 1.2 and 1.4

This section is devoted to the proof of Theorems 1.2 and 1.4. We will use some ideas introduced by Bahri in [1] and some technical estimates. We start by the following proposition which gives a parametrization of the function  $u_\lambda$ . It follows from corresponding statements in [2,3].

**Proposition 3.1.** *The following minimization problem*

$$\min\{\|u_\lambda - \alpha_1 P\delta_{(a_1, \lambda_1)} + \alpha_2 P\delta_{(a_2, \lambda_2)}\|, \alpha_i > 0, \lambda_i > 0, a_i \in \Omega\}$$

*has a unique solution  $(\alpha_1, \alpha_2, a_1, a_2, \lambda_1, \lambda_2)$ . In particular, we can write  $u_\lambda$  as follows*

$$u_\lambda = \alpha_1 P\delta_{(a_1, \lambda_1)} - \alpha_2 P\delta_{(a_2, \lambda_2)} + v,$$

*where  $v \in H_0^1(\Omega)$  such that*

$$(V_0) : \langle v, \varphi \rangle = 0, \quad \varphi \in \{P\delta_{(a_i, \lambda_i)}, \partial P\delta_{(a_i, \lambda_i)}/\partial \lambda_i, \partial P\delta_{(a_i, \lambda_i)}/\partial a_i^j, i = 1, 2, 1 \leq j \leq n\}, \quad (3.1)$$

*where  $a_i^j$  denotes the  $j$ th component of  $a_i$ .*

**Remark 3.2.** For each  $i = 1, 2$ , the point  $a_i$  is close to  $a_{\lambda,i}$  and for each parameter  $\lambda_i$  we have that  $\lambda_i/\mu_{\lambda,i}$  is close to 1, where  $a_{\lambda,i}$ ,  $\mu_{\lambda,i}$  are defined in Theorem 1.1 and  $a_i$ ,  $\lambda_i$  are defined in Proposition 3.1.

As usual in this type of problems, we first deal with the  $v$ -part of  $u_\lambda$ , in order to show that it is negligible with respect to the concentration phenomenon.

**Lemma 3.3.** *The function  $v$  defined in Proposition 3.1 satisfies the following estimate*

$$\|v\| \leq c \begin{cases} \sum_i \frac{\lambda}{\lambda_i^{(n-2)/2}} + \frac{1}{(\lambda_i d_i)^{n-2}} + \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{(n-2)/n} & \text{if } n = 4, 5, \\ \sum_i \lambda \frac{\log \lambda_i}{\lambda_i^2} + \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} + \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{2/3} & \text{if } n = 6, \\ \sum_i \frac{\lambda}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{(n+2)/2}} + \varepsilon_{12}^{(n+2)/2(n-2)} (\log \varepsilon_{12}^{-1})^{(n+2)/2n} & \text{if } n > 6, \end{cases}$$

*where  $\varepsilon_{12}$  is defined by  $\varepsilon_{12} = (\lambda_1/\lambda_2 + \lambda_2/\lambda_1 + \lambda_1 \lambda_2 |a_{\lambda,1} - a_{\lambda,2}|^2)^{(2-n)/2}$ .*



**Proof.** For the sake of simplicity, we will write  $P\delta_i$  instead of  $P\delta_{(a_i, \lambda_i)}$ .

Since  $u_\lambda = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$  is a solution of (1) and  $v$  is orthogonal to  $P\delta_i$ , we obtain

$$\begin{aligned} \int_{\Omega} -\Delta u_\lambda v &= \|v\|^2 = \int_{\Omega} |u_\lambda|^{4/(n-2)} u_\lambda v + \lambda \int_{\Omega} u_\lambda v \\ &= \int_{\Omega} |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2) v + \frac{n+2}{n-2} \int_{\Omega} |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}} v^2 \\ &\quad + O(\|v\|^{\inf(3, 2n/(n-2))}) + \lambda \int_{\Omega} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2) v + \lambda \int_{\Omega} v^2. \end{aligned}$$

Thus we derive

$$Q(v, v) = f(v) + O(\|v\|^{\inf(3, 2n/(n-2))}), \quad (3.2)$$

where

$$\begin{aligned} Q(v, v) &= \|v\|^2 - \frac{n+2}{n-2} \int_{\Omega} |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}} v^2 - \lambda \int_{\Omega} v^2 \quad \text{and} \\ f(v) &= \int_{\Omega} |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2) v + \lambda \int_{\Omega} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2) v. \end{aligned} \quad (3.3)$$

As in [1], it is easy to see that  $Q$  is positive definite on

$$E := E_{(a_1, a_2, \lambda_1, \lambda_2)} := \{u \in H_0^1(\Omega) \mid u \text{ satisfies } (V_0)\},$$

that means, there exists  $\beta_0 > 0$  satisfying

$$Q(w, w) \geq \beta_0 \|w\|^2 \quad \text{for each } w \in E.$$

Therefore, from (3.2) we get

$$\|v\| = O(\|f\|). \quad (3.4)$$

It remains to estimate  $\|f\|$ . To this aim, we need to compute

$$\begin{aligned} &\int_{\Omega} |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2) v \\ &= \int_{\Omega} (\alpha_1 P\delta_1)^{\frac{n+2}{n-2}} v - \int_{\Omega} (\alpha_2 P\delta_2)^{\frac{n+2}{n-2}} v \\ &\quad + O\left(\int_{\Omega} (\delta_1 \delta_2)^{\frac{n+2}{2(n-2)}} |v| + \sum_{i \neq j} \int_{\Omega} \delta_i^{\frac{4}{n-2}} \delta_j |v| \quad (\text{if } n < 6)\right). \end{aligned} \quad (3.5)$$

Since  $v \in E$ , we get

$$\begin{aligned} \int_{\Omega} P \delta_i^{\frac{n+2}{n-2}} v &= \int_{\Omega} \delta_i^{\frac{n+2}{n-2}} v + O\left(\int_{\Omega} \delta_i^{\frac{4}{n-2}} (\delta_i - P\delta_i) |v|\right) \\ &\leq c \|v\| \begin{cases} \frac{1}{(\lambda_i d_i)^{n-2}} & \text{if } n = 4, 5, \\ \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} & \text{if } n = 6, \\ \frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}} & \text{if } n > 6. \end{cases} \end{aligned} \quad (3.6)$$

For the other integrals of the right-hand side of (3.5), using Hölder's inequality, we derive

$$\int_{\Omega} (\delta_1 \delta_2)^{(n+2)/2(n-2)} |v| \leq c \|v\| \varepsilon_{12}^{(n+2)/2(n-2)} (\log \varepsilon_{12}^{-1})^{(n+2)/2n}, \quad (3.7)$$

and if  $n < 6$ , we have  $4/(n-2) > 1$  and therefore

$$\int_{\Omega} \delta_i^{4/(n-2)} \delta_j |v| \leq c \|v\| \left( \int_{\Omega} (\delta_1 \delta_2)^{n/(n-2)} \right)^{(n-2)/n} \leq c \|v\| \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{(n-2)/n}. \quad (3.8)$$

Now, we need to estimate the second integral in (3.3). Using Hölder's inequality, we get

$$\int_{\Omega} P \delta_i |v| \leq c \|v\| \begin{cases} \frac{1}{\lambda_i^2} & \text{if } n > 6, \\ \frac{\log \lambda_i}{\lambda_i^2} & \text{if } n = 6, \\ \frac{1}{\lambda_i^{(n-2)/2}} & \text{if } n = 4, 5. \end{cases} \quad (3.9)$$

Combining (3.3)–(3.9), the proof of Lemma 3.3 follows.  $\square$

Now, we need to estimate the coefficients  $\alpha_i$  which are defined in Proposition 3.1.

**Proposition 3.4.** *Let  $n \geq 4$ . Each coefficient  $\alpha_i$ , defined in Proposition 3.1, satisfies the following estimate*

$$1 - \alpha_i^{4/(n-2)} = \begin{cases} O\left(\frac{1}{(\lambda_i d_i)^{n-2}} + \varepsilon_{12} + \|v\| + \frac{\lambda_i}{\lambda_i^2}\right) & \text{if } n \geq 5, \\ O\left(\frac{1}{(\lambda_i d_i)^{n-2}} + \varepsilon_{12} + \|v\| + \lambda \frac{\log(\lambda_i)}{\lambda_i^2}\right) & \text{if } n = 4. \end{cases}$$

**Proof.** Multiplying (1) by  $P\delta_1$  and integrating on  $\Omega$ , we obtain

$$\alpha_1 \int_{\Omega} \delta_1^p P \delta_1 - \alpha_2 \int_{\Omega} \delta_2^p P \delta_1 = \int_{\Omega} |u_{\lambda}|^{p-1} u_{\lambda} P \delta_1 + \lambda \int_{\Omega} u_{\lambda} P \delta_1, \quad (3.10)$$

where  $p = (n + 2)/(n - 2)$ . Observe that, using [1], we get

$$\int_{\Omega} \delta_1^p P \delta_1 = S^{n/2} + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right), \quad \int_{\Omega} \delta_2^p P \delta_1 = O(\varepsilon_{12}), \quad (3.11)$$

where  $S$  is the best Sobolev constant for the embedding  $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ .

For the last integral of (3.10), it is easy to obtain

$$\begin{aligned} \int_{\Omega} P \delta_2 P \delta_1 &= O(\varepsilon_{12}), \quad \int_{\Omega} |v| P \delta_1 \leq c \|v\|; \\ \int_{\Omega} P \delta_1^2 &\leq \begin{cases} \frac{c}{\lambda_1^2} & \text{if } n \geq 5, \\ c \frac{\log(\lambda_1)}{\lambda_1^2} & \text{if } n = 4. \end{cases} \end{aligned} \quad (3.12)$$

It remains to estimate the third term in (3.10).

$$\begin{aligned} \int_{\Omega} |u_{\lambda}|^{p-1} u_{\lambda} P \delta_1 &= \alpha_1^{\frac{n+2}{n-2}} \int_{\Omega} P \delta_1^{\frac{2n}{n-2}} + O\left(\int_{\Omega} \delta_1^{\frac{n+2}{n-2}} (\delta_2 + |v|) + \int_{\Omega} (\delta_2^{\frac{n+2}{n-2}} + |v|^{\frac{n+2}{n-2}}) \delta_1\right) \\ &= \alpha_1^{\frac{n+2}{n-2}} S^{n/2} + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}} + \varepsilon_{12} + \|v\|\right). \end{aligned} \quad (3.13)$$

Combining (3.10)–(3.13), the proof of Proposition 3.4 follows for  $i = 1$  and in the same way, Proposition 3.4 is true for  $i = 2$ .  $\square$

Next, we prove two crucial propositions which give some estimates involving the points  $a_i$  and the variables  $\lambda_i$ .

**Proposition 3.5.** (a) For  $n \geq 5$ , we have the following estimate

$$\begin{aligned} c_1 \frac{n-2}{2} \frac{H(a_i, a_i)}{\lambda_i^{n-2}} - c_1 \left( \lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_1, a_2)}{(\lambda_1 \lambda_2)^{(n-2)/2}} \right) - \lambda \frac{c_2}{\lambda_i^2} &= A_i \quad \text{with} \\ A_i &= O\left(\sum_{k=1,2} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{12}^{n/(n-2)} \log \varepsilon_{12}^{-1} + \lambda \varepsilon_{12} + \frac{\lambda}{(\lambda_i d_i)^{n-2}} + \lambda^2 R_i^2\right), \end{aligned}$$

where  $i \in \{1, 2\}$ ,  $c_1 = \beta_n \int_{\mathbb{R}^n} \delta_{(0,1)}^{(n+2)/(n-2)}$ ,  $c_2 = \frac{n-2}{2} \beta_n^2 \int_{\mathbb{R}^n} \frac{|y|^2 - 1}{(1+|y|^2)^{n-1}}$ , and  $R_i$  satisfies

$$R_i = \begin{cases} O\left(\frac{1}{\lambda_i^2}\right) & \text{if } n > 6, \\ O\left(\frac{\log \lambda_i}{\lambda_i^2}\right) & \text{if } n = 6, \\ O\left(\frac{1}{\lambda_i^{(n-2)/2}}\right) & \text{if } n \leq 5. \end{cases} \quad (3.14)$$

(b) For  $n = 4$ , if we assume that  $d_i := d(a_i, \partial\Omega) \geq d_0$ ,  $i = 1, 2$ , and  $|a_1 - a_2| \geq d_0$  for a fixed constant  $d_0 > 0$ , then we have

$$c_1 \frac{n-2}{2} \frac{H(a_i, a_i)}{\lambda_i^{n-2}} - c_1 \left( \lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_1, a_2)}{(\lambda_1 \lambda_2)^{(n-2)/2}} \right) - \lambda c_3 \frac{\log(\lambda_i)}{\lambda_i^2} = A_i,$$

where  $c_3 = \beta_4^2 \omega_4$ , with  $\omega_4$  denoting the area of the unit sphere of  $\mathbb{R}^4$ .

**Proof.** We will prove Proposition 3.5 for  $i = 1$  and for  $i = 2$ , the same holds. Multiplying (1) by  $\lambda_1 \partial P \delta_1 / \partial \lambda_1$  and integrating on  $\Omega$ , we obtain

$$\alpha_1 \int_{\Omega} \delta_1^p \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} - \alpha_2 \int_{\Omega} \delta_2^p \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = \int_{\Omega} |u_{\lambda}|^{p-1} u_{\lambda} \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} + \lambda \int_{\Omega} u_{\lambda} \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1}, \quad (3.15)$$

where  $p = (n+2)/(n-2)$ . Let  $d_i = d(a_i, \partial\Omega)$ . Using [1], we derive

$$\int_{\Omega} \delta_1^p \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = \frac{n-2}{2} c_1 \frac{H(a_1, a_1)}{\lambda_1^{n-2}} + O\left(\frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^n}\right), \quad (3.16)$$

$$\int_{\Omega} \delta_2^p \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = c_1 \left( \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + \frac{n-2}{2} \frac{H(a_1, a_2)}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}}} \right) + R, \quad (3.17)$$

where  $R$  satisfies

$$R = O\left(\sum_{k=1,2} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{12}^{\frac{n}{n-2}} \log \varepsilon_{12}^{-1}\right). \quad (3.18)$$

For the last integral in (3.15), using the fact that  $\lambda_1 |\partial P \delta_1 / \partial \lambda_1| \leq c \delta_1$ , we have

$$\left| \int_{\Omega} v \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} \right| \leq c \|v\| \begin{cases} \frac{\log \lambda_1}{\lambda_1^2} & \text{if } n = 6, \\ \frac{1}{\lambda_1^{\inf(2, (n-2)/2)}} & \text{if } n \neq 6, \end{cases} \quad (3.19)$$

$$\left| \int_{\Omega} P \delta_2 \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} \right| \leq c \int_{\Omega} \delta_1 \delta_2 \leq c \varepsilon_{12}, \quad (3.20)$$

$$\int_{\Omega} P \delta_1 \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = \int_{\Omega} \delta_1 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) = -\frac{c_2}{\lambda_1^2} + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right), \quad (3.21)$$

where we have used in the last equation the fact that  $n \geq 5$  and  $|\delta_1 - P \delta_1|_{L^\infty} \leq c(\lambda_1 d_1^2)^{(2-n)/2}$ .

We deal now with the third integral of (3.15). Observe that

$$\begin{aligned} \int_{\Omega} |u_{\lambda}|^{\frac{4}{n-2}} u_{\lambda} \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} &= \int_{\Omega} |\alpha_1 P \delta_1 - \alpha_2 P \delta_2|^{\frac{4}{n-2}} (\alpha_1 P \delta_1 - \alpha_2 P \delta_2) \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} \\ &\quad + \frac{n+2}{n-2} \int_{\Omega} |\alpha_1 P \delta_1 - \alpha_2 P \delta_2|^{\frac{4}{n-2}} v \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} + O(\|v\|^2 + \varepsilon_{12}^{\frac{n}{n-2}} \log \varepsilon_{12}^{-1}). \end{aligned} \quad (3.22)$$

For the last integral in (3.22), we write

$$\begin{aligned} &\int_{\Omega} |\alpha_1 P \delta_1 - \alpha_2 P \delta_2|^{\frac{4}{n-2}} v \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} \\ &= \int_{\Omega} (\alpha_1 P \delta_1)^{\frac{4}{n-2}} v \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} + O\left(\int_{\Omega \setminus A} P \delta_2^{\frac{4}{n-2}} P \delta_1 |v| + \int_A P \delta_1^{\frac{4}{n-2}} P \delta_2 |v|\right), \end{aligned} \quad (3.23)$$

where  $A = \{x \mid 2\alpha_2 P \delta_2 \leq \alpha_1 P \delta_1\}$ . Observe that, for  $n \geq 6$ , we have  $4/(n-2) \leq 1$ , thus

$$\begin{aligned} \int_{\Omega \setminus A} P \delta_2^{\frac{4}{n-2}} P \delta_1 |v| + \int_A P \delta_1^{\frac{4}{n-2}} P \delta_2 |v| &\leq c \int |v| (\delta_1 \delta_2)^{(n+2)/2(n-2)} \\ &\leq c \|v\| \varepsilon_{12}^{(n+2)/2(n-2)} (\log \varepsilon_{12}^{-1})^{(n+2)/2n}. \end{aligned} \quad (3.24)$$

But for  $n \leq 5$ , we have

$$\int_{\Omega \setminus A} P \delta_2^{4/(n-2)} P \delta_1 |v| + \int_A P \delta_1^{4/(n-2)} P \delta_2 |v| \leq c \|v\| \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{(n-2)/n}. \quad (3.25)$$

For the other integral in (3.23), using [1], we have

$$\int_{\Omega} P \delta_1^{\frac{4}{n-2}} v \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = \begin{cases} O\left(\|v\| \frac{1}{(\lambda_1 d_1)^{\inf(n-2, (n+2)/2)}}\right) & \text{if } n \neq 6, \\ O\left(\|v\| \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^4}\right) & \text{if } n = 6. \end{cases} \quad (3.26)$$

It remains to estimate the second integral of (3.22). It is easy to obtain

$$\begin{aligned} &\int_{\Omega} |\alpha_1 P \delta_1 - \alpha_2 P \delta_2|^{\frac{4}{n-2}} (\alpha_1 P \delta_1 - \alpha_2 P \delta_2) \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} \\ &= \int_{\Omega} (\alpha_1 P \delta_1)^{\frac{n+2}{n-2}} \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} - \int_{\Omega} (\alpha_2 P \delta_2)^{\frac{n+2}{n-2}} \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} - \frac{n+2}{n-2} \int_{\Omega} \alpha_2 P \delta_2 (\alpha_1 P \delta_1)^{\frac{4}{n-2}} \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} \\ &\quad + O(\varepsilon_{12}^{\frac{n}{n-2}} \log \varepsilon_{12}^{-1}). \end{aligned} \quad (3.27)$$

Now, using [1], we have

$$\int_{\Omega} P \delta_1^p \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = (n-2)c_1 \frac{H(a_1, a_1)}{\lambda_1^{n-2}} + O\left(\frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^n}\right), \quad (3.28)$$

$$\int_{\Omega} P \delta_2^p \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = c_1 \left( \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + \frac{n-2}{2} \frac{H(a_1, a_2)}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}}} \right) + R, \quad (3.29)$$

$$\frac{n+2}{n-2} \int_{\Omega} P \delta_2 P \delta_1^{\frac{4}{n-2}} \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = c_1 \left( \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + \frac{n-2}{2} \frac{H(a_1, a_2)}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}}} \right) + R, \quad (3.30)$$

where  $R$  is defined by (3.18). Therefore, using (3.15)–(3.30), Lemma 3.3 and Proposition 3.4, the proof of claim (a) of Proposition 3.5 follows.

To prove claim (b), observe that we have used the fact that  $n \geq 5$  only in (3.21). Then we need to compute

$$\int_{\Omega} P \delta_1 \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = \int_{\Omega} \delta_1 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} + O\left(\frac{1}{(\lambda_1 d_1)^2}\right).$$

Since  $\Omega$  is bounded and we have assumed that  $d_1 \geq d_0$ , we have  $B(a_1, d_0) \subset \Omega \subset B(a_1, R)$  for a fixed  $R > 0$ . Hence

$$\int_{\Omega} \delta_1 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} \leq \beta_4^2 \frac{\omega_4}{\lambda_1^2} \int_0^{\lambda_1 d_0} \frac{1-r^2}{(1+r^2)^3} r^3 dr = -\beta_4^2 \frac{\omega_4}{\lambda_1^2} (\log(\lambda_1) + O(1)).$$

In the same way

$$\int_{\Omega} \delta_1 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} \geq \beta_4^2 \frac{\omega_4}{\lambda_1^2} \int_0^{\lambda_1 R} \frac{1-r^2}{(1+r^2)^3} r^3 dr = -\beta_4^2 \frac{\omega_4}{\lambda_1^2} (\log(\lambda_1) + O(1)).$$

Hence

$$\int_{\Omega} P \delta_1 \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} = -\beta_4^2 \frac{\omega_4}{\lambda_1^2} \log(\lambda_1) + O\left(\frac{1}{\lambda_1^2}\right).$$

The proof of claim (b) follows.  $\square$

**Proposition 3.6.** *Let  $n \geq 4$  and assume that  $\lambda_1$  and  $\lambda_2$  are of the same order. We have the following estimate*

$$\begin{aligned} & \frac{1}{\lambda_i^{n-1}} \frac{\partial H(a_i, a_i)}{\partial a_i} + \frac{2}{\lambda_i} \left( \frac{\partial \varepsilon_{12}}{\partial a_i} - \frac{\partial H}{\partial a_i}(a_1, a_2) \frac{1}{(\lambda_1 \lambda_2)^{(n-2)/2}} \right) \\ &= \begin{cases} O\left(\sum \frac{1}{(\lambda_k d_k)^n} + \varepsilon_{12}^{n/(n-2)} \log \varepsilon_{12}^{-1} + \frac{\lambda}{\lambda_i} \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{n-2}{n}} + \frac{\lambda}{(\lambda_i d_i)^{n-1}} + \lambda^2 R_i^2\right) & \text{if } n \geq 5, \\ o\left(\sum \frac{1}{(\lambda_k \gamma)^3} + \varepsilon_{12}^{3/2}\right) & \text{if } n = 4, \end{cases} \end{aligned}$$

where  $i \in \{1, 2\}$ ,  $\gamma := (1/2) \min(d_1, d_2, |a_1 - a_2|)$  and  $R_i$  is defined in (3.14).

**Proof.** First, we deal with the dimension  $n \geq 5$ . The proof is similar to the proof of Proposition 3.5. But there exist some integrals which have different estimate. We will focus on those integrals. In fact, Eqs. (3.15), (3.19), (3.22)–(3.27) are also true if we change  $\lambda_1 \partial P \delta_1 / \partial \lambda_1$  by  $(1/\lambda_1) \partial P \delta_1 / \partial a_1$ . It remains to deal with the other equations:

$$\int_{\Omega} \delta_1^p \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} = \frac{1}{2} \frac{c_1}{\lambda_1^{n-1}} \frac{\partial H(a_1, a_1)}{\partial a_1} + O\left(\frac{1}{(\lambda_1 d_1)^n}\right), \quad (3.31)$$

$$\begin{aligned} \int_{\Omega} \delta_2^p \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} &= \frac{c_1}{\lambda_1} \left( \frac{\partial \varepsilon_{12}}{\partial a_1} - \frac{1}{(\lambda_1 \lambda_2)^{(n-2)/2}} \frac{\partial H}{\partial a_1}(a_1, a_2) \right) \\ &+ O\left(\sum_{k=1,2} \frac{1}{(\lambda_k d_k)^n} + \lambda_2 |a_1 - a_2| \varepsilon_{12}^{(n+1)/(n-2)}\right). \end{aligned} \quad (3.32)$$

Using [1,14], we have  $|\delta_1 - P \delta_1| \leq c(\lambda_1 d_1^2)^{(2-n)/2}$  and  $|\partial(\delta_1 - P \delta_1)/\partial a_1| \leq c\lambda_1^{(2-n)/2} d_1^{(1-n)}$ . Furthermore, an easy computation shows  $|\partial \delta_1 / \partial a_1| = c\delta_1^{n/(n-2)} \lambda_1 |x - a_1|$ . Thus,

$$\begin{aligned} \int_{\Omega} P \delta_1 \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} &= \int_{\Omega \setminus B_1} \delta_1 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} - \int_{\Omega} (\delta_1 - P \delta_1) \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} - \int_{\Omega} P \delta_1 \frac{1}{\lambda_1} \frac{\partial}{\partial a_1} (\delta_1 - P \delta_1) \\ &= O\left(\frac{1}{(\lambda_1 d_1)^{n-1}}\right), \end{aligned} \quad (3.33)$$

where we have used the evenness of  $\delta_1$  and the oddness of its derivative and where  $B_1 = B(a_1, d_1)$ .

$$\begin{aligned} \int P \delta_2 \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} &= \int P \delta_2 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} - P \delta_2 \frac{1}{\lambda_1} \frac{\partial}{\partial a_1} (\delta_1 - P \delta_1) \\ &= O\left(\int (\delta_2 \delta_1) \delta_1^{\frac{2}{n-2}} |x - a_1| + \left\| \frac{1}{\lambda_1} \frac{\partial}{\partial a_1} (\delta_1 - P \delta_1) \right\| \left( \int \delta_2^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}}\right) \\ &= O\left(\frac{1}{\lambda_1} \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{n-2}{n}} + \frac{1}{(\lambda_1 d_1)^{n/2}} R_2\right), \end{aligned}$$

where  $R_2$  is defined in (3.14) and where we have used Hölder's inequality and [1,14].

$$\int P \delta_1^{\frac{n+2}{n-2}} \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} = -\frac{c_1}{\lambda_1^{n-1}} \frac{\partial H(a_1, a_1)}{\partial a_1} + O\left(\frac{1}{(\lambda_1 d_1)^n}\right), \quad (3.34)$$

$$\int P \delta_2^p \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} = \int_{\Omega} \delta_2^p \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \log \varepsilon_{12}^{-1} + \sum_{k=1,2} \frac{1}{(\lambda_k d_k)^n}\right), \quad (3.35)$$

$$P \int_{\Omega} P \delta_2 P \delta_1^{p-1} \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} = \int_{\Omega} \delta_2^p \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \log \varepsilon_{12}^{-1} + \sum_{k=1,2} \frac{1}{(\lambda_k d_k)^n}\right), \quad (3.36)$$

where  $p = (n+2)/(n-2)$ .

The proof of Proposition 3.6 is completed for the higher dimensions,  $n \geq 5$ .

Now, for  $n = 4$ , we have  $4/(n - 2) = 2$ . Let  $\gamma := (1/2) \min(d_1, d_2, |a_1 - a_2|)$  and  $B_\gamma^i := B(a_i, \gamma)$ . Observe that  $B_\gamma^1 \cap B_\gamma^2 = \emptyset$  and for  $x \in B_\gamma^1$ , we have  $\delta_2(x) \leq (\lambda_2 \gamma^2)^{-1}$ . Expanding  $u_\lambda^3$ , we need to compute the following integrals:

$$\begin{aligned} \int_{\Omega} P \delta_2 v P \delta_1 \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} &= \int_{\Omega} P \delta_2 v P \delta_1 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} + O\left(\frac{\|v\|}{(\lambda_1 d_1)^2}\right) \\ &\leq \frac{c}{\lambda_2 \gamma^2} \int_{B_\gamma^1} |v| \delta_1^3 |x - a_1| + O\left(\frac{\|v\|}{(\lambda_1 \gamma)^2}\right) = O\left(\frac{\|v\|}{(\lambda_1 \gamma)^2}\right). \end{aligned}$$

In the same way, we have

$$\begin{aligned} \int_{\Omega} P \delta_2^2 v \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} &\leq \frac{c}{(\lambda_2 \gamma^2)^2} \int_{B_\gamma^1} |v| \delta_1^2 |x - a_1| + \frac{c}{\lambda_1^2 \gamma^3} \int_{B_\gamma^2} |v| \delta_2^2 + O\left(\frac{\|v\|}{(\lambda_1 \gamma)^2}\right) = O\left(\frac{\|v\|}{(\lambda_1 \gamma)^2}\right), \\ \left| \int v \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} \right| &\leq c \int_{B_\gamma^1 \cup \Omega \setminus B_\gamma^1} |v| \delta_1^2 |x - a_1| + \frac{c \|v\|}{(\lambda_1 d_1)^2} \leq c \|v\| \left( \frac{1}{(\lambda_1 \gamma)^2} + \frac{\log^{3/4}(\lambda \gamma)}{\lambda_1^2} \right), \\ \left| \int P \delta_2 \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} \right| &\leq \frac{c}{\lambda_2 \gamma^2} \int_{B_\gamma^1} \delta_1^2 |x - a_1| + \frac{c}{\lambda_1 \gamma} \int_{\Omega \setminus B_\gamma^1} \delta_1 \delta_2 \leq \frac{c}{\lambda_1^2 \lambda_2 \gamma^2} + \varepsilon_{12} \frac{c}{\lambda_1 \gamma}, \\ \int (|v|^3 + P \delta_2 v^2 + P \delta_1 P \delta_2^2) \delta_1 &= O(\|v\|^3 + \varepsilon_{12}^2 \log \varepsilon_{12}^{-1}), \\ \int_{\Omega} v^2 P \delta_1 \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial a_1} &= \int_{B_\gamma^1} v^2 \delta_1 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} + O\left(\frac{\|v\|^2}{\lambda_1 d_1}\right). \end{aligned}$$

To estimate the last integral, we use an original idea due to Rey [15], namely, we write

$$v = v_1 + v_2 + w,$$

where  $v_i$  denotes the projection of  $v$  onto  $H_0^1(B_\gamma^i)$ , that is,

$$\Delta v_i = \Delta v \quad \text{in } B_\gamma^i; \quad v_i = 0 \quad \text{on } \partial B_\gamma^i.$$

Observe that  $v_i$  can be assumed to be defined in all  $\Omega$  since it can be continued by 0 in  $\Omega \setminus B_\gamma^i$ . We have

$$v = v_i + w \quad \text{in } B_\gamma^i, \quad \text{with } \Delta w = 0 \text{ in } B_\gamma^i.$$

We split  $v_i$  in an even part  $v_i^e$  and an odd part  $v_i^o$  with respect to  $(x - a_i)_k$ , thus we have

$$v = v_i^e + v_i^o + w \quad \text{in } B_\gamma^i \quad \text{with } \Delta w = 0 \text{ in } B_\gamma^i.$$



Following [15], we prove that

$$\|v^o\| \leq c \frac{\|v\|}{\lambda_1 \gamma} \quad \text{and} \quad \left| \int_{B_\gamma^1} \delta_1 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_k} (2v - w) w \right| \leq c \|v\| \left( \frac{1}{(\lambda_1 \gamma)^4} + \frac{\|v\|}{\lambda_1 \gamma} + \|v\|^3 \right).$$

Hence, using the evenness of  $v_1^e$  and the oddness of  $v_1^o$ , we obtain for  $k = 1, \dots, 4$ ,

$$\int_{B_\gamma^1} v^2 \delta_1 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_k} = \int_{B_\gamma^1} \delta_1 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_k} (2v - w) w + O(\|v_1^o\| \|v_1^e\|) = o\left(\frac{1}{(\lambda_1 \gamma)^3}\right). \quad (3.37)$$

The proof follows from (3.31), (3.32), (3.34)–(3.36) and the above estimates. The proof of Proposition 3.6 is thereby completed.  $\square$

Now, we are going to use the above propositions in order to prove that the concentration points are far away from the boundary and the distance between them is bounded away from zero.

**Lemma 3.7.** *There exists a constant  $c_0 > 0$ , independent of  $\lambda$ , such that*

$$\begin{aligned} \text{(i)} \quad c_0 &\leq \frac{|a_1 - a_2|}{d_i} \leq c_0^{-1}, \quad \text{for } i = 1, 2; \\ \text{(ii)} \quad c_0 &\leq \frac{d_1}{d_2} \leq c_0^{-1}. \end{aligned}$$

**Proof.** From Proposition 3.5, it is easy to obtain, for  $n \geq 5$ , the following estimate

$$\frac{\lambda}{\lambda_i^2} = O\left(\sum \frac{1}{(\lambda_j d_j)^{n-2}} + \varepsilon_{12}\right), \quad \text{for } i = 1, 2. \quad (3.38)$$

Since we have assumed that  $-\max u_\lambda / \min u_\lambda$  is bounded above and below, we derive that  $\lambda_1$  and  $\lambda_2$  are of the same order. This implies

$$\varepsilon_{12} = \frac{1}{(\lambda_1 \lambda_2 |a_1 - a_2|^2)^{(n-2)/2}} + O(\varepsilon_{12}^{n/(n-2)}). \quad (3.39)$$

Without loss of generality, we can assume that  $d_1 \leq d_2$ .

Regarding claim (i), arguing by contradiction, in a first step, we assume that  $d_1 = o(|a_1 - a_2|)$ . In this case, it is easy to obtain

$$\begin{aligned} \left| \frac{\partial \varepsilon_{12}}{\partial a_1} - \frac{1}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}}} \frac{\partial H}{\partial a_1}(a_1, a_2) \right| &\leq \frac{1}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}}} \left( \frac{c}{|a_1 - a_2|^{n-1}} + \frac{c}{d_1 |a_1 - a_2|^{n-2}} \right) \\ &= o\left(\frac{1}{d_1 (\lambda_1 d_1)^{n-2}}\right). \end{aligned} \quad (3.40)$$

Since  $d_1 \rightarrow 0$ , we have  $\partial H(a_1, a_1)/\partial n_1 \sim c/d_1^{n-1}$  (see [13]), where  $n_1$  denotes the outward normal vector to  $\Omega_{d_1} := \{x: d(x, \partial\Omega) > d_1\}$  at  $a_1$ . Thus, Proposition 3.6 and (3.40) give a contradiction and we derive that  $d_1/|a_1 - a_2|$  is bounded below.

It remains to prove that  $d_1/|a_1 - a_2|$  is bounded above. To this aim, we argue by contradiction and we assume that  $|a_1 - a_2| = o(d_1)$ . Therefore,

$$\begin{aligned} \frac{1}{\lambda_1^{n-1}} \left| \frac{\partial H(a_1, a_1)}{\partial a_1} \right| + \frac{1}{\lambda_1(\lambda_1\lambda_2)^{(n-2)/2}} \left| \frac{\partial H(a_1, a_2)}{\partial a_1} \right| &\leq \frac{c}{(\lambda_1 d_1)^{n-1}} \\ &= o\left(\frac{1}{(\lambda_1\lambda_2|a_1 - a_2|^2)^{(n-1)/2}}\right). \end{aligned} \quad (3.41)$$

Regarding the other term in Proposition 3.6, it satisfies

$$\left| \frac{1}{\lambda_1} \frac{\partial \varepsilon_{12}}{\partial a_1} \right| \sim \frac{n-2}{\lambda_1(\lambda_1\lambda_2)^{(n-2)/2}} \frac{1}{|a_1 - a_2|^{n-1}} \geq \frac{c}{(\lambda_1\lambda_2|a_1 - a_2|^2)^{(n-1)/2}}. \quad (3.42)$$

Hence, (3.41), (3.42) and Proposition 3.6 give a contradiction, and therefore  $|a_1 - a_2|/d_1$  is bounded below. Thus, there exists a constant  $c$  such that

$$cd_1 \leq |a_1 - a_2| \leq c^{-1}d_1 \leq c^{-1}d_2.$$

It remains to prove that  $|a_1 - a_2| \geq cd_2$ . Observe that, if  $2d_1 \geq d_2$ , we are done. If not, it is easy to obtain that  $|a_1 - a_2| \geq d_2 - d_1 \geq d_2/2$ . Hence claim (i) is proved.

Regarding claim (ii), it follows immediately from claim (i). Hence the proof of the proposition is completed.  $\square$

Now, we will prove that the concentration points are in a compact set of  $\Omega$  and each point  $a_i$  converges to  $\bar{a}_i$  satisfying  $\bar{a}_1 \neq \bar{a}_2$  which implies the first part of Theorem 1.2.

**Lemma 3.8.** *There exists a positive constant  $d_0$  such that*

$$|a_1 - a_2| \geq d_0; \quad d_i \geq d_0 \quad \text{for } i = 1, 2.$$

**Proof.** We need only to prove that  $d_1 \rightarrow 0$  and using Lemma 3.7, we derive the other assertions. Arguing by contradiction, we assume that  $d_1 \rightarrow 0$ . Observe that Lemma 3.7 implies that  $d_1$ ,  $d_2$  and  $|a_1 - a_2|$  are of the same order. Thus, we can use the behavior of the functions  $G$  and  $H$  near the boundary which are given in [4,5]. Let  $n_i$  be the outward normal vector to  $\Omega_{d_i} := \{x: d(x, \partial\Omega) > d_i\}$  at  $a_i$ . We then have

$$\frac{\partial H(a_i, a_i)}{\partial n_i} = \frac{n-2}{2^{n-2}d_i^{n-1}}(1 + o(1)), \quad (3.43)$$

$$\frac{\partial G(a_i, a_j)}{\partial n_i} = -(n-2) \frac{d_j - d_i}{|a_1 - a_2|^n} - \frac{(n-2)(d_1 + d_2)}{(|a_1 - a_2|^2 + 4d_1d_2)^{n/2}} + o\left(\frac{1}{d_i^{n-1}}\right). \quad (3.44)$$

Using Proposition 3.6, we derive

$$\sum_{i=1,2} \frac{n-2}{2^{n-2} \lambda_i^{n-2} d_i^{n-1}} - \frac{4(n-2)}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}}} \frac{d_1 + d_2}{(|a_1 - a_2|^2 + 4d_1 d_2)^{\frac{n}{2}}} = o\left(\sum \frac{1}{\lambda_i^{n-2} d_i^{n-1}}\right). \quad (3.45)$$

Now we prove that the second term in (3.45) is dominated by the first one. In fact, since  $|a_1 - a_2| \geq |d_1 - d_2|$ , we get  $(d_1 + d_2)^2 \leq |a_1 - a_2|^2 + 4d_1 d_2$  and therefore

$$\begin{aligned} \frac{4}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}}} \frac{d_1 + d_2}{(|a_1 - a_2|^2 + 4d_1 d_2)^{\frac{n}{2}}} &\leq \frac{4}{(\lambda_1 \lambda_2)^{\frac{n-2}{2}} (|a_1 - a_2|^2 + 4d_1 d_2)^{\frac{n-1}{2}}} \\ &\leq \frac{4}{(c_0^2 + 4)^{\frac{n-1}{2}} (\lambda_1^{n-2} d_1^{n-1})^{\frac{1}{2}} (\lambda_2^{n-2} d_2^{n-1})^{\frac{1}{2}}} \\ &\leq \frac{2}{(c_0^2 + 4)^{(n-1)/2}} \sum_{i=1,2} \frac{1}{\lambda_i^{n-2} d_i^{n-1}}, \end{aligned} \quad (3.46)$$

where we have used Lemma 3.7. Since

$$\frac{1}{2^{n-2}} - \frac{2}{(c_0^2 + 4)^{(n-1)/2}} \geq c' > 0,$$

we derive a contradiction from (3.45) and (3.46). Hence our lemma follows.  $\square$

Now, we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Lemmas 3.7 and 3.8 show that each  $a_i$  converges to  $\bar{a}_i \in \Omega$  with  $\bar{a}_1 \neq \bar{a}_2$ .

Now we will prove the second part of Theorem 1.2. Observe that, from Propositions 3.5, 3.6 and the fact that  $\lambda_1$  and  $\lambda_2$  are of the same order, we have

$$\frac{H(a_i, a_i)}{\lambda_i^{n-2}} + \frac{G(a_1, a_2)}{(\lambda_1 \lambda_2)^{(n-2)/2}} - \frac{c_2}{c_1} \frac{2}{n-2} \frac{\lambda}{\lambda_i^2} = o\left(\frac{1}{\lambda_i^{n-2}}\right) \quad \text{if } n \geq 5, \quad (3.47)$$

$$\frac{H(a_i, a_i)}{\lambda_i^2} + \frac{G(a_1, a_2)}{\lambda_1 \lambda_2} - \frac{c_3}{c_1} \lambda \frac{\log(\lambda_i)}{\lambda_i^2} = o\left(\frac{1}{\lambda_i^2}\right) \quad \text{if } n = 4, \quad (3.48)$$

$$\frac{1}{\lambda_i^{n-2}} \frac{\partial H(a_i, a_i)}{\partial a_i} + \frac{2}{(\lambda_1 \lambda_2)^{(n-2)/2}} \frac{\partial G(a_1, a_2)}{\partial a_i} = o\left(\frac{1}{\lambda_i^{n-2}}\right). \quad (3.49)$$

Let us introduce the following change of variable

$$\frac{1}{\lambda_i^{(n-2)/2}} = \Lambda_i \left( \frac{c_2}{c_1} \lambda \right)^{(n-2)/(2n-8)} \quad \text{if } n \geq 5; \quad \lambda \log(\lambda_i) = \frac{c_1}{c_3} \Lambda_i \quad \text{if } n = 4.$$

Note that, (3.47) and (3.49) imply, for  $i = 1, 2$  and  $j \neq i$ ,

$$H(a_i, a_i)\Lambda_i + G(a_1, a_2)\Lambda_j - \frac{2}{n-2}\Lambda_i^{(6-n)/(n-2)} = o(\Lambda_i), \quad (3.50)$$

$$\frac{\partial H(a_i, a_i)}{\partial a_i}\Lambda_i^2 + 2\frac{\partial G(a_1, a_2)}{\partial a_i}\Lambda_1\Lambda_2 = o(\Lambda_i^2). \quad (3.51)$$

Recall that, we have proved that each  $a_i$  converges to  $\bar{a}_i \in \Omega$  with  $\bar{a}_1 \neq \bar{a}_2$ . Thus the functions  $H$ ,  $G$  and its derivatives are bounded. Therefore, from (3.47), it is easy to obtain that

$$\frac{c'}{\lambda_i^{n-2}} \leq \frac{\lambda}{\lambda_i^2} \leq \frac{c}{\lambda_i^{n-2}},$$

and therefore, for each  $i = 1, 2$ ,  $\Lambda_i$  is bounded above and below. Hence, each  $\Lambda_i$  converges to  $\bar{\Lambda}_i > 0$  (up to a sequence). This implies (1.6).

Now, passing to the limit in (3.50) and (3.51), we get

$$H(\bar{a}_i, \bar{a}_i)\bar{\Lambda}_i + G(\bar{a}_1, \bar{a}_2)\bar{\Lambda}_j - \frac{2}{n-2}\bar{\Lambda}_i^{(6-n)/(n-2)} = 0, \quad (3.52)$$

$$\frac{\partial H(\bar{a}_i, \bar{a}_i)}{\partial \bar{a}_i}\bar{\Lambda}_i^2 + 2\frac{\partial G(\bar{a}_1, \bar{a}_2)}{\partial \bar{a}_i}\bar{\Lambda}_1\bar{\Lambda}_2 = 0, \quad (3.53)$$

where  $i = 1, 2$  and  $j \neq i$ .

Equations (3.52) and (3.53) imply that  $\nabla\psi(\bar{a}_1, \bar{a}_2, \bar{\Lambda}_1, \bar{\Lambda}_2) = 0$ . Hence  $(\bar{a}_1, \bar{a}_2, \bar{\Lambda}_1, \bar{\Lambda}_2)$  is a critical point of  $\psi$ . The proof of Theorem 1.2 is thereby completed for  $n \geq 5$ .

If  $n = 4$ , denoting  $\gamma_i = \lambda_i/\lambda_j$ , with  $j \neq i$ , then (3.48) and (3.49) imply

$$H(a_i, a_i) + \gamma_i G(a_1, a_2) - \Lambda_i = o(1), \quad (3.54)$$

$$\frac{\partial H(a_i, a_i)}{\partial a_i} + 2\gamma_i \frac{\partial G(a_1, a_2)}{\partial a_i} = o(1). \quad (3.55)$$

From (1.5), we derive that  $\gamma_i$  converges to a constant  $\bar{\gamma}_i$ , with  $\bar{\gamma}_1^{-1} = \bar{\gamma}_2 := \bar{\gamma}$  (up to a subsequence). Furthermore, since  $\bar{a}_i \in \Omega$  and  $\bar{a}_1 \neq \bar{a}_2$ , using (3.54) we get that  $\Lambda_i$  is bounded above and below, for  $i = 1, 2$ . Thus, up to a subsequence,  $\Lambda_i$  converges to a constant  $\bar{\Lambda}_i$  and it is easy to prove that  $\bar{\Lambda}_1 = \bar{\Lambda}_2 := \bar{\Lambda}$ . This implies (1.7).

Now, passing to the limit in (3.54) and (3.55), we get

$$H(\bar{a}_i, \bar{a}_i) + \bar{\gamma}_i G(\bar{a}_1, \bar{a}_2) - \bar{\Lambda} = 0, \quad (3.56)$$

$$\frac{\partial H(\bar{a}_i, \bar{a}_i)}{\partial \bar{a}_i} + 2\bar{\gamma}_i \frac{\partial G(\bar{a}_1, \bar{a}_2)}{\partial \bar{a}_i} = 0. \quad (3.57)$$

This ends the proof of Theorem 1.2.  $\square$

We end this section by proving Theorem 1.4.

**Proof of Theorem 1.4.** Observe that, by Theorem 1.1, we know that  $u_\lambda$  can be written as

$$u_\lambda = P\delta_{(a_{\lambda,1}, \mu_{\lambda,1})} - P\delta_{(a_{\lambda,2}, \mu_{\lambda,2})} + v,$$

with  $\|v\| \rightarrow 0$ ,  $u_\lambda(a_{\lambda,1}) = \max u_\lambda$ ,  $u_\lambda(a_{\lambda,2}) = \min u_\lambda$  and  $\mu_{\lambda,i}|a_{\lambda,1} - a_{\lambda,2}| \rightarrow \infty$ , for  $i = 1, 2$ . Furthermore, the concentration speeds satisfy (1.6) and (1.7).

Let  $h_\lambda := \max d(x, \mathcal{S})^{(n-2)/2} |u_\lambda(x)|$  where  $\mathcal{S} = \{a_{\lambda,1}, a_{\lambda,2}\}$ . It is easy to prove that  $h_\lambda$  is bounded (if not, we can construct another blow-up point and therefore the energy of  $u_\lambda$  becomes bigger than  $3S^{n/2}$  which gives a contradiction).

Let  $d_{\lambda,1} = d(a_{\lambda,1}, \partial\Omega_+)$  and  $d_{\lambda,2} = d(a_{\lambda,2}, \partial\Omega_-)$ . We need to prove that  $d_{\lambda,i} \rightarrow 0$  as  $\lambda \rightarrow 0$ . Arguing by contradiction, assume that  $d_{\lambda,1} \leq d_{\lambda,2}$  and  $d_{\lambda,1} \rightarrow 0$ . We define the following function

$$w_\lambda(X) := d_{\lambda,1}^{(n-2)/2} u_\lambda(a_{\lambda,1} + d_{\lambda,1}X) \quad \text{for } X \in \Omega'_{\lambda,+} := d_{\lambda,1}^{-1}(\Omega_+ - a_{\lambda,1}).$$

Observe that  $B(0,1) \subset \Omega'_{\lambda,+}$  and  $w_\lambda > 0$  in  $\Omega'_{\lambda,+}$ . Since  $h_\lambda$  is bounded and  $d_{\lambda,1}^{(n-2)/2} u_\lambda(a_{\lambda,1}) \rightarrow \infty$ , we derive that 0 is an isolated blow-up point of  $(w_\lambda)$ . Now we can proceed as in [6, Appendix] to prove that 0 is an isolated simple blow-up point of  $(w_\lambda)$ . In fact, the proof of this assertion is almost the same as in [6, Proposition 5.9]. The only argument which is different is to prove (5.25) of [6]. For  $n \geq 4$ , we need to change the proof. Observe that, as in [6, Proposition 5.7], if 0 is an isolated simple blow-up point of  $(w_\lambda)$  then we have

$$w_\lambda(y) \leq c w_\lambda(0) \left(1 + \beta_n^{2/(2-n)} w_\lambda(0)^{4/(n-2)} |y|^2\right)^{(2-n)/2}, \quad \text{for } |y| \leq 1/2,$$

where  $c$  is a positive constant independent of  $\lambda$ . Therefore (5.25) of [6] follows from (1.6) and (1.7).

Then, we proceed as in the proof of (3.13) of [6] to conclude that  $d_{\lambda,1} \rightarrow 0$  which completes the proof of the first part of Theorem 1.4 that is  $\bar{a}_1$  and  $\bar{a}_2$  are isolated simple blow-up points of  $(u_\lambda)$ .

Finally, we follow the proof of (4.10) of [6] to complete the proof of our theorem.  $\square$

#### 4. Proof of Theorem 1.6

Let us start by proving the first statement of the theorem.

**Lemma 4.1.** *Let  $(u_\lambda)$  be as in Theorem 1.6. Then for  $\lambda$  close to 0 if  $n \geq 4$  or  $\lambda$  close to  $\bar{\lambda}(\Omega)$  if  $n = 3$ , the concentration points  $a_{\lambda,i}$  of  $u_\lambda$ , given by Theorem 1.1, lay on the same line passing through the origin.*

**Proof.** Observe that, for  $n \geq 4$ , we have many information about the limit concentration points  $\bar{a}_i$  which simplify the proof. Hence we will start by proving Lemma 4.1 for the higher dimensions.

(i) **The case  $n \geq 4$ .** For sake of simplicity, we assume that  $\Omega$  is the unit ball. In this case, the Green function and its regular part are given by

$$G(x, y) = \frac{1}{|x - y|^{n-2}} - \frac{1}{(|x|^2|y|^2 + 1 - 2\langle x, y \rangle)^{(n-2)/2}}, \quad (4.1)$$

$$H(x, x) = \frac{1}{(1 - |x|^2)^{n-2}}. \quad (4.2)$$

It is easy to see that

$$\frac{\partial H(x, x)}{\partial x} = \frac{2(n-2)x}{(1-|x|^2)^{n-1}}, \quad (4.3)$$

$$\frac{\partial G}{\partial x}(x, y) = \frac{(n-2)(y-x)}{|x-y|^n} - \frac{(n-2)(y-|y|^2x)}{(|x|^2|y|^2 + 1 - 2\langle x, y \rangle)^{n/2}}. \quad (4.4)$$

Thus, (3.53), (3.57), (4.3) and (4.4) imply that the concentration points  $\bar{a}_1, \bar{a}_2$  are different from the origin. Thus, for  $\lambda$  close to 0, we have that  $a_{\lambda,i}$  is far away from the origin, for  $i = 1, 2$ .

Now we will prove that the limit concentration points  $\bar{a}_i$  lie on the same line passing through the origin. To this aim, we can assume, without loss of generality, that  $\bar{a}_1$  lies on the  $x_n$ -axis and therefore  $\bar{a}_1 = (0, \dots, 0, \gamma_1)$ . Again, (3.53), (3.57), (4.3) and (4.4) imply that the  $j$ th component of  $\bar{a}_2$ , for  $j = 1, \dots, n-1$ , has to vanish. Hence,  $\bar{a}_2$  lies also on the  $x_n$ -axis, that means,  $\bar{a}_2 = (0, \dots, 0, \gamma_2)$ .

We now prove that  $\gamma_1\gamma_2 < 0$ . To this aim, we will study the sign of the derivative of  $G$  with respect to the  $n$ th component of  $\bar{a}_1$  and  $\bar{a}_2$ . Assume that  $\gamma_1 \geq \gamma_2$ , thus an easy computation shows that

$$\begin{aligned} \frac{\partial G}{\partial (\bar{a}_2)_n}(\bar{a}_1, \bar{a}_2) &= \frac{n-2}{(\gamma_1 - \gamma_2)^{n-1}} - \frac{(n-2)\gamma_1}{(1 - \gamma_1\gamma_2)^{n-1}} > 0, \\ \frac{\partial G}{\partial (\bar{a}_1)_n}(\bar{a}_1, \bar{a}_2) &= -\frac{n-2}{(\gamma_1 - \gamma_2)^{n-1}} - \frac{(n-2)\gamma_2}{(1 - \gamma_1\gamma_2)^{n-1}} < 0, \end{aligned}$$

since  $1 - \gamma_1\gamma_2 > \gamma_1 - \gamma_2$  and  $|\gamma_i| < 1$ , for  $i = 1, 2$ . Thus, from (3.53), (3.57) and (4.3), we derive that  $\gamma_1 > 0$  and  $\gamma_2 < 0$ .

Now, let us assume that the line connecting  $a_{\lambda,1}$  with the origin is the  $x_n$ -axis and  $a_{\lambda,1}$  lies on the half-space given by the condition  $\{x_n > 0\}$ . From the above arguments, we derive that the concentration points  $a_{\lambda,1}$  and  $a_{\lambda,2}$  lay on different sides with respect to the hyperplane  $\{x \in \mathbb{R}^n: x_n = 0\}$ . Then we assume, by contradiction, that for a sequence  $\lambda_k \rightarrow 0$ , the points  $a_{\lambda_k,2} := a_{k,2}$  are given by  $a_{k,2} = (\alpha_k, x_2^k, \dots, x_n^k)$ ,  $\alpha_k > 0$ , where the first coordinate  $\alpha_k$  represents the distance of  $a_{k,2}$  from the  $x_n$ -axis. Observe that, without loss of generality, we can assume that  $d(a_{k,1}, 0) \leq d(a_{k,2}, 0)$  if 0 is the origin.

We consider the half-ball  $B_1^- = \{x \in \Omega: x_1 < 0\}$  and we claim that

$$w_k(x) = u_k(x) - v_k(x) \geq 0 \quad \text{in } B_1^-, \quad (4.5)$$

where  $v_k$  is the reflected function of  $u_k$  with respect to the hyperplane  $T_1 = \{x \in \mathbb{R}^n: x_1 = 0\}$ .

Before proving (4.5) let us show that if (4.5) holds we get a contradiction. Indeed, since  $w_k \not\equiv 0$  in  $B_1^-$ , because  $a_{k,2}$  is not on the symmetry hyperplane, by the strong maximum principle (4.5) implies  $w_k > 0$  in  $B_1^-$ . Then, applying the Hopf lemma at the point  $a_{k,1}$ , which is on the symmetry hyperplane  $T_1$ , we get

$$0 > \frac{\partial w_k}{\partial x_1}(a_{k,1}) = -2 \frac{\partial u_k}{\partial x_1}(a_{k,1}) \quad (4.6)$$

which is a contradiction, because  $a_{k,1}$  is a critical point of  $u_k$ .

To prove (4.5) let us consider the balls  $D_{1,k} = B(a_{k,1}, R\eta_k)$  and  $D_{2,k} = B(a_{k,2}, R\eta_k)$  with  $R > 1$  to be fixed later and where  $\eta_k = (u_k(a_{k,1}))^{-2/(n-2)}$ .

Then we take  $\theta_{0,k} \in [0, \pi/2]$  and consider the hyperplane  $T_{\theta_{0,k}} = \{x = (x_1, \dots, x_n) : x_1 \sin \theta_{0,k} + x_n \cos \theta_{0,k} = 0\}$ . We observe that we can choose  $\theta_{0,k} < \pi/2$  and close to  $\pi/2$  such that the balls  $D_{i,k}$  lay on different sides with respect to the hyperplane  $T_{\theta_{0,k}}$ . Note that this is always possible because  $d(a_{k,1}, 0)/\eta_k \rightarrow +\infty$  (by Theorem 1.2). Then, denoting by  $D'_{i,k}$  the reflections of  $D_{i,k}$  with respect to  $T_{\theta_{0,k}}$ , we have that  $w_{k,\theta_{0,k}}(x) = u_k - v_{k,\theta_{0,k}} > 0$  for any  $x \in D_{1,k} \cup D'_{2,k}$ , where  $v_{k,\theta_{0,k}}$  is the reflected function of  $u_k$  with respect to the hyperplane  $T_{\theta_{0,k}}$ . Hence in the domain  $B_{\theta_{0,k}}^- \setminus (D_{1,k} \cup D'_{2,k})$  with  $B_{\theta_{0,k}}^- = \{x \in \Omega : x_1 \sin \theta_{0,k} + x_n \cos \theta_{0,k} < 0\}$  we have that  $w_{k,\theta_{0,k}}$  satisfies

$$\begin{cases} -\Delta w_{k,\theta_{0,k}} - c_k w_{k,\theta_{0,k}} = 0 & \text{in } B_{\theta_{0,k}}^- \setminus (D_{1,k} \cup D'_{2,k}), \\ w_{k,\theta_{0,k}} \geq 0 & \text{on } \partial(B_{\theta_{0,k}}^- \setminus (D_{1,k} \cup D'_{2,k})), \end{cases} \quad (4.7)$$

where

$$c_k(x) = \int_0^1 (2^* - 1) [tu_k(x) + (1-t)v_{k,\theta_{0,k}}(x)]^{2^*-2} + \lambda dt. \quad (4.8)$$

Then since  $u_k$  concentrates only at  $a_{k,i}$  it is easy to see (see [8,10]) that it is possible to choose  $R$  such that, for  $k$  sufficiently large, the first eigenvalue of the linear operator  $L_k = -\Delta - c_k$  in  $B_{\theta_{0,k}}^- \setminus (D_{1,k} \cup D'_{2,k})$  with zero Dirichlet boundary condition is positive (if  $n = 3$ , we require that  $\lambda < \lambda_1(\Omega)$ ). This implies, by (4.7), that  $w_{k,\theta_{0,k}} \geq 0$  in  $B_{\theta_{0,k}}^- \setminus (D_{1,k} \cup D'_{2,k})$  and hence in the whole  $B_{\theta_{0,k}}^-$ . Moreover, by the maximum principle we get

$$w_{k,\theta_{0,k}} > 0 \quad \text{in } B_{\theta_{0,k}}^- \text{ for } k \text{ sufficiently large.} \quad (4.9)$$

Now we fix  $k$  and rotate the hyperplane  $T_\theta := \{x : x_1 \sin \theta + x_n \cos \theta = 0\}$  varying  $\theta \in [\theta_{0,k}, \pi/2]$  with condition (4.9) and consider

$$\tilde{\theta} = \sup\{\theta \in [\theta_{0,k}, \pi/2] : w_{k,\theta} > 0 \text{ in } B_\theta^-\},$$

with  $B_\theta^- := \{x \in \Omega : x_1 \sin \theta + x_n \cos \theta < 0\}$ . A standard argument, based on the maximum principle (see [8]) shows that  $\tilde{\theta} = \pi/2$ , i.e., (4.5) holds. The proof of Lemma 4.1 is thereby completed for the dimension  $n \geq 4$ .

(ii) **The case  $n = 3$ .** Obviously we can assume that none of the points  $a_{\lambda,i}$  is the center of the ball, otherwise the statement is trivially true.

As in the above case, let us assume that the line connecting  $a_{\lambda,1}$  with the origin is the  $x_n$ -axis, and  $a_{\lambda,1}$  lies on the half-space given by the condition  $\{x_n > 0\}$ . Then we assume, by contradiction, that for a sequence  $\lambda_k \rightarrow \bar{\lambda}(\Omega)$  the points  $a_{\lambda_k,2} := a_{k,2}$  are given by  $a_{k,2} = (\alpha_k, x_2^k, \dots, x_n^k)$ ,  $\alpha_k > 0$ , where the first coordinate  $\alpha_k$  represents the distance of  $a_{k,2}$  from the  $x_n$ -axis. We define  $\eta_k := (u_k(a_{k,1}))^{-2}$  and consider two cases.

*Case 1.* Here we assume that the points  $a_{k,1}$  and  $a_{k,2}$  lay on different sides with respect to the hyperplane  $\{x \in \mathbb{R}^n : x_n = 0\}$ , i.e.,  $x_n^k < 0$ . Without loss of generality we can assume that  $d(a_{k,1}, 0) \leq d(a_{k,2}, 0)$ .

In this case, if  $d(a_{k,1}, 0)/\eta_k \rightarrow \infty$ , arguing as before we get (4.5). Hence we derive a contradiction as before.

In the case when  $d(a_{k,1}, 0)/\eta_k \rightarrow \ell \geq 0$ , we get (4.5) arguing as in the next case, claim 2.

*Case 2.* Here we assume that the points  $a_{k,1}$  and  $a_{k,2}$  lay on the same side with respect to the hyperplane  $\{x \in \mathbb{R}^n: x_n = 0\}$  and, without loss of generality, we can assume that  $d(a_{k,1}, 0) \geq d(a_{k,2}, 0)$ . We complete the proof in three steps.

*Claim 1.* It is not possible that

$$\alpha_k/\eta_k \rightarrow \infty. \quad (4.10)$$

Assume that (4.10) holds. Then, arguing as in the first case ( $n \geq 4$ ), we get (4.5) in the half-ball  $B_1^-$ , which gives a contradiction. Indeed, by (4.10) and (1.5) we can choose  $\theta_{0,k} \in [0, \pi/2]$  such that the balls  $D_{1,k}$  and  $D_{2,k}$  defined before, lay on different sides with respect to the hyperplane  $T_{\theta_{0,k}}$ . Then the proof to get (4.5) is exactly the same.

*Claim 2.* It is not possible that

$$\alpha_k/\eta_k \rightarrow \ell > 0. \quad (4.11)$$

Let us assume (4.11) and consider  $\theta_k \in [0, \pi/2]$  such that the points  $a_{k,1}$  and  $a_{k,2}$  lay on different sides with respect to the hyperplane  $T_{\theta_k} := \{x \in \mathbb{R}^n: x_1 \sin \theta_k + x_n \cos \theta_k = 0\}$  and have the same distance  $d_k > 0$  from this hyperplane. Of course, because of (4.11) we have

$$d_k/\eta_k \rightarrow \ell_1 > 0, \quad \text{as } k \rightarrow \infty. \quad (4.12)$$

Again we choose the two balls  $D_{1,k}$  and  $D_{2,k}$  as in the case  $n \geq 4$ . If, for  $k$  large, the two balls lay on different sides with respect to the hyperplane  $T_{\theta_k}$  we can prove (4.5) exactly in the same way as before. Otherwise we observe that we have, for  $k$  large

$$u_k(x) \geq v_{k,\theta_k}(x) \quad \text{in the sets } E_{1,k}^{\theta_k} := B_{\theta_k}^- \cap D_{1,k} \text{ and } E_{2,k}^{\theta_k} := B_{\theta_k}^- \cap D'_{2,k}. \quad (4.13)$$

Indeed, if (4.13) does not hold for example in  $E_{1,k}^{\theta_k}$  we could construct a sequence of points  $x_k \in E_{1,k}^{\theta_k}$  such that

$$u_k(x_k) < v_{k,\theta_k}(x_k). \quad (4.14)$$

Then there would exist a sequence of points  $\xi_k \in E_{1,k}^{\theta_k}$  such that

$$\frac{\partial u_k}{\partial \theta_k}(\xi_k) < 0. \quad (4.15)$$

Thus, by rescaling  $u_k$  in the usual way around  $a_{n_k,1}$ , and using (4.12) we would get a point  $\xi \in (E_{\ell_1}^{\theta_0})^- := \{x \in \mathbb{R}^n: x_1 \sin \theta_0 + x_n \cos \theta_0 < -\ell_1 < 0\}$  such that  $(\partial u / \partial \theta_0)(\xi) \leq 0$  while  $\partial u / \partial \theta_0 > 0$  in  $(E_{\ell_1}^{\theta_0})^-$ ,  $\theta_0$  being the limit of  $\theta_k$ . Hence (4.13) holds.

Then, in the domain  $B_k^- \setminus (E_{1,k}^{\theta_k} \cup E_{2,k}^{\theta_k})$  we have that  $w_k$  satisfies a problem analogous to (4.7) with the same coefficient  $c_k(x)$ . Thus, arguing as in the case  $n \geq 4$ , we get the analogous of (4.8) in  $B_{\theta_k}^-$  for  $k$  large and hence (4.5). Therefor (4.11) cannot hold.

*Claim 3.* It is not possible that

$$\alpha_k/\eta_k \rightarrow 0. \quad (4.16)$$



The proof is exactly the same as that of the analogous claim of [8, Lemma 6, Claim 3]. This ends the proof of Lemma 4.1.  $\square$

Now, we are in the position to prove our theorem.

**Proof of Theorem 1.6.** The first part of the statement is exactly Lemma 4.1. The proof that  $u_\lambda$  is symmetry with respect to any hyperplane passing through the axis containing  $a_{\lambda,1}$  and  $a_{\lambda,2}$  is the same as that of [8, Theorem 2] with some obvious modification due to the fact that  $u_\lambda(a_{\lambda,1})$  and  $u_\lambda(a_{\lambda,2})$  have different sign, so we omit it. Now we are going to prove the last part of the statement of Theorem 1.6. Let  $T$  be an hyperplane passing through the origin but not containing  $a_{\lambda,1}$  and let  $D^+$  be the half-ball containing  $a_{\lambda,1}$  and determined by  $T$  and the ball  $\Omega$ . Let  $w_\lambda = u_\lambda - v_\lambda$ ,  $v_\lambda$  being the reflection of  $u_\lambda$  with respect to  $T$ . We observe that  $w_\lambda = 0$  on  $\partial D^+$ ,  $w_\lambda(a_{\lambda,1}) > 0$  and  $w_\lambda(\tilde{a}_{\lambda,2}) > 0$ , where  $\tilde{a}_{\lambda,2}$  is the reflection of  $a_{\lambda,2}$  with respect to  $T$ . Since  $u_\lambda$  concentrates at  $a_{\lambda,1}$  and  $a_{\lambda,2}$ , we can choose  $\alpha > 0$  such that  $w_\lambda > 0$  on  $F_i := \partial B(a_{\lambda,i}, \alpha)$  and  $w_\lambda$  is small in  $D^+ \setminus (F_1 \cup F_2)$ . Thus applying the maximum principle we get  $w_\lambda > 0$  in  $D^+ \setminus (F_1 \cup F_2)$ . Using again the fact that  $u_\lambda$  concentrates at  $a_{\lambda,1}$  and  $a_{\lambda,2}$ , we see that  $w_\lambda > 0$  in  $D^+$ . Thus the last statement of Theorem 1.6 is an easy consequence of Hopf's lemma. This ends the proof of our result.  $\square$

## References

- [1] A. Bahri, Critical Point at Infinity in Some Variational Problems, Pitman Res. Notes Math. Ser., vol. 182, Longman Sci. Tech., Harlow, 1989.
- [2] A. Bahri, An invariant for Yamabe-type flows with applications to scalar curvature problems in high dimension, *Duke Math. J.* 81 (1996) 323–466.
- [3] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of topology of the domain, *Comm. Pure Appl. Math.* 41 (1988) 255–294.
- [4] A. Bahri, Y.Y. Li, O. Rey, On a variational problem with lack of compactness: The topological effect of the critical points at infinity, *Calc. Var. Partial Differential Equations* 3 (1995) 67–94.
- [5] M. Ben Ayed, H. Chtioui, M. Hammami, A Morse lemma at infinity for Yamabe-type problems in domains, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 20 (2003) 543–577.
- [6] M. Ben Ayed, K. El Mehdi, F. Pacella, Blow-up and nonexistence of sign-changing solutions to the Brezis–Nirenberg problem in dimension three, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, in press.
- [7] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical exponents, *Comm. Pure Appl. Math.* 36 (1983) 437–477.
- [8] D. Castorina, F. Pacella, Symmetry of positive solutions of an almost critical problem in an annulus, *Calc. Var. Partial Differential Equations* 23 (2005) 125–138.
- [9] M. Clapp, T. Weth, Multiple solutions for the Brezis–Nirenberg problem, *Adv. Differential Equations* 10 (2005) 463–480.
- [10] K. El Mehdi, F. Pacella, Morse index and blow-up points of solutions of some nonlinear problems, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 13 (2002) 101–105.
- [11] Y.Y. Li, Prescribing scalar curvature on  $S^n$  and related topics, Part I, *J. Differential Equations* 120 (1995) 319–410.
- [12] A.M. Micheletti, A. Pistoia, On the effect of the domain geometry on the existence of sign-changing solutions to elliptic problems with critical and supercritical growth, *Nonlinearity* 17 (2004) 1–16.
- [13] O. Rey, A multiplicity result for a variational problem with lack of compactness, *Nonlinear Anal.* 13 (1989) 1241–1249.
- [14] O. Rey, The role of Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent, *J. Funct. Anal.* 89 (1990) 1–52.
- [15] O. Rey, The topological impact of critical points at infinity in a variational problem with lack of compactness: The dimension 3, *Adv. Differential Equations* 4 (1999) 581–616.
- [16] M. Struwe, A global compactness result for elliptic boundary value problem involving limiting nonlinearities, *Math. Z.* 187 (1984) 511–517.